Abstract Algebra: Questions Teachers Refused to Answer in High School

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Universal Algebra

- Set
- Operation (unary, binary, n-ary)

$\mathbb{N}$ $\mathbb{R}[x]$ (Q2)
$\mathbb{Z}$ $\mathbb{R}^2$ or $\mathbb{R}^3$ → hard
$\mathbb{Q}$ (Q1) (Q3)
$\mathbb{R}$

Group-like (1 set, 1 op)

Ex: $(\mathbb{Z}/12\mathbb{Z}, +)$ (0, inv, comm/assoc) [am v pm] abelian group

Ex: $(S_3, \circ)$ (relabel s.t. each label used once) nonabelian group

$(1\ 2\ 3)$

$(1\ 2)(2\ 3) = (1\ 2\ 3)$

$(2\ 3)(1\ 2) = (1\ 3\ 2) \equiv (3\ 2\ 1)$ inverses!

freq implicit operation

SNote: weaker cond (release ... general notion)

Ex: $\mathbb{N} = \{1, 2, 3, \dots\}$ semigroup

$\mathbb{Z}^+ = \{0, 1, 2, 3, \dots\}$ monoid

Magma Aside

Ex: $(\mathbb{R}^3, \times)$ Cross Product

$(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \hat{0}$

$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath}$

Non associative

Lost All Properties!

The Rest of Group Theory:

- Substructures
- Construction
- Classification
- unary/generalized unary op (homomorphisms)

Ring-like (1 set, 2 op)

Ex: $\mathbb{R}$, $\mathbb{Q}$ field

- caveat about zero WHY?

$$0x = 0 \quad \forall x$$

Pf: $\underset{\text{id \#1, op 2}}{0 \cdot x} = \underset{\text{op 1 ident}}{(0+0) \cdot x} = \underset{\underline{\text{dist}}}{0x + 0x} \Rightarrow 0x = 0$ by inverse

See, real question is how op "play" w/ one another

Ex: $\mathbb{R}[x]$ Comm Unitary Ring

$(\mathbb{R}[x])^{\times}$:= all units (ele w/ inverses) = $\mathbb{R}$

[Matrices are an example of a division ring (w/ ele in F)]

Q2: Solving Systems of Eqns (matrice hint)

$\frac{1}{4}x^2 + \frac{3}{4}x = -\frac{1}{2}$

$\downarrow$

$x^2 + 3x + 2 = 0$

$(x+1)(x+2)$

$\not\downarrow$

$x^3 + 3x^2 + 2x = 0$

$x(x+1)(x+2) = 0$

$x^3 - 2x^2 = x$

$\downarrow$

$x(x^2 - 2x - 1)$

$x(x-1)^2$

$\not\updownarrow$

$x^2 - 2x = 1$

$x^2 - 2x - 1 = (x-1)^2$

* Factorization unique up to multiplication by units

$10(x-1) = 0 \equiv (x-1) = 0$ "divide by 10" (stupid thought)

Module-like

$\mathbb{R}^3$ over $\mathbb{R}$ v.s.

$\mathbb{R}^3$ over $\mathbb{Z}$ module

$M_n(\mathbb{R})$ algebra

Why just addn and mult?

D $\quad a(b+c) = ab + ac$ "Linear v Bilinear"

D' $\quad a \cdot (bc) = (a \cdot b)c = b(a \cdot c)$ | $a \cdot (b+c) = a \cdot b + c = b + a \cdot c$

Ex: $7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix}$

D → "0" and "1" disjoint

D' → "0" and "1" identified gen identities

... under special circumstances

* only 2 op w/ ident distribute D (else 0 = 1)

↳ 3 op if we distribute D' on later (not really on a single set)

Ring of Fractions

$$\frac{1}{3} + \frac{1}{4} \overset{?}{=} \frac{2}{7} \overset{?}{=} \frac{1}{12} = \frac{7}{12}$$ why this? "Criss-Cross Apple-Sauce"

Gen Desire:

- $\mathbb{Z}$ act on $\mathbb{Q}$ - classify up to units
- "real world"

$$\boxed{\frac{a}{b} = \frac{c}{d} \iff ad = bc}$$

This forces

$$\frac{a}{d} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b}\frac{c}{d} = \frac{ac}{bd} \quad \forall \text{ rings}$$

* power lies in generalization

My Notation:

set → D ← operation

D 1-1	$r(v+w) = rv + rw$	linear
D' 1-2	$r(v \cdot w) = (rv)w$	bilinear
D 2-1	$(r+s)v = rv + sv$	linear
D' 2-2	$(rs)v = r(sv)$	bilinear

⇝ one should wonder

n-ary operation → 2^n distributive types

? release restriction

? all realised

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Basic Outline of Universal Algebra

Structure	Set 1									-				Set 2								
	Op 1				-	Op 2				Action				Op 1				-	Op 2			
	A	Id	In	C	D	A	Id	In	C	D1-1	D’1-2	D2-1	D’2-2	A	Id	In	C	D	A	Id	In	C
Magma				b																		
Semigroup	x			b																		
Monoid	x	x		b																		
Group	x	x	x																			
Abelian Group	x	x	x	x																		
Ring	x	x	x	x	x	x			b													
Unitary Ring	x	x	x	x	x	x	x		b													
Division Ring	x	x	x	x	x	x	x	x	b													
Field	x	x	x	x	x	x	x	x	x													
Module	x	x	x	x						x		x	x	x	x	x	x	x	x			
Vector Space	x	x	x	x						x		x	x	x	x	x	x	x	x	x	x	x
Algebra	x	x	x	x	x	x				x	x	x	x	x	x	x	x	x	x	x	x	x