The Matrix: A Mathematical Construct

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3 Perspectives on Matrices:

1. Sums

$$(\alpha+\beta)(a+b) = \begin{matrix} \alpha a + \alpha b \\ \beta a + \beta b \end{matrix}$$

Shorthand: ① drop plus signs

$$\begin{bmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{bmatrix}$$

2. Linear Transformations (Maps)

$y = mx + b$ (affine)

$y = mx$ Implicit Choice

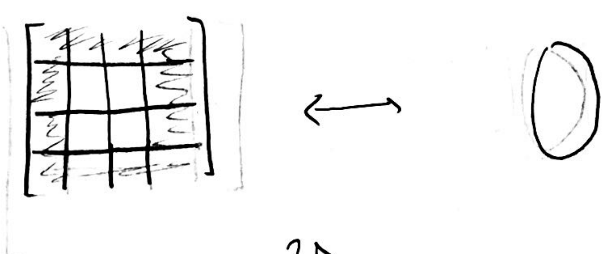
Arb Basis $\rightarrow$ N-dim

$\vec{v} = c_1 \alpha + c_2 \beta$

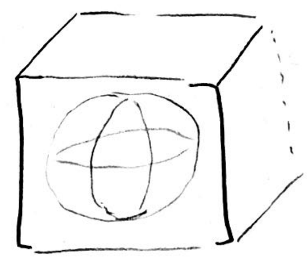
Shorthand: ② drop basis

$$\vec{v} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

3. Pictoral (Pixels) – comp sci



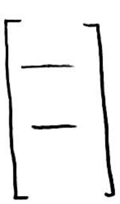
2D



3 D

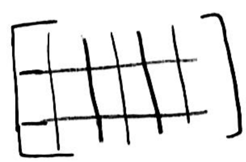
Rank – # of axes

Dimension – # entries on in axis



Rank 1

Dim 3



Rank 2

Dim 3 x 2

Note: order convention in Rank 2 lost in higher dim

Motivation

$$(c_1\alpha + c_2\beta)(d_1 a + d_2 b) \xrightarrow{\text{don't know}} \begin{bmatrix} c_1 d_1 & c_1 d_2 \\ c_2 d_1 & c_2 d_2 \end{bmatrix}$$ Shorthand ①+②

$$\xrightarrow{\text{know}} c_1 d_1 + c_2 d_2 \quad \text{iff} \quad \begin{array}{l} a\alpha = 1 \\ b\beta = 1 \\ a\beta = b\alpha = 0 \end{array}$$

How does (α, β) "play" w/ (a, b)?

$$\begin{array}{lll} c_1 d_1 \alpha a & + & c_1 d_2 \alpha b \\ c_2 d_1 \beta a & + & c_2 d_2 \beta b \end{array} \qquad M = \begin{bmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{bmatrix}$$ "Metric Tensor"

Matrix Formalism:

$$[c_1 \;\; c_2] \underbrace{\begin{bmatrix} \alpha a & \alpha b \\ \beta a & \beta b \end{bmatrix}}_{M^\mu{}_\nu} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Colm

$$[c_1 \;\; c_2]\left(d_1 \underbrace{\begin{bmatrix} \alpha a \\ \beta a \end{bmatrix}}_{a} + d_2 \underbrace{\begin{bmatrix} \alpha b \\ \beta b \end{bmatrix}}_{b}\right) = \underbrace{[c_1 \;\; c_2]}_{\alpha \quad \beta} \begin{bmatrix} d_1(\alpha a) + d_2(\alpha b) \\ d_1(\beta a) + d_2(\beta b) \end{bmatrix}$$

Row Colm

$$[c_1 \;\; c_2] \begin{bmatrix} 1^{st} \text{ row} & 1^{st} \text{ colm} \\ 2^{nd} \text{ row} & 1^{st} \text{ colm} \end{bmatrix}^T \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

... multiply out $c_1 d_1 \alpha a + c_1 d_2 \alpha b + c_2 d_1 \beta a + c_2 d_2 \beta b$

Hidden b/c gen work w/ $M = Id$

__Dual__ - (a, b), (α, β) s.t. have well defined metric b/w them

Transformation Properties

Contravariant - opposes Δ of variable

Covariant - follows Δ of variable

Ex: "divided axis"

$10 \, \frac{m}{s} \rightarrow 1000 \, \frac{cm}{s}$ $\qquad$ $10 \, m \rightarrow 1000 \, cm$

$10 \, \frac{J}{m} \rightarrow \frac{1}{10} \, \frac{J}{cm}$

Shorthand: ③ drop components

T^{μ} $\qquad$ Same symbol $\Rightarrow$ dual

T_{μ}

"Einstein Notation"

dx $\qquad$ contra

$\frac{d}{dx}$ $\qquad$ cov

Tensor Formalism:

Rank n $\quad (r, s)$

$\uparrow$ contra "column" $\quad \uparrow$ cov "row"

Ex:

$T_{\mu}{}^{\nu}$ $\qquad$ Rank 2 $\;(1,1)$

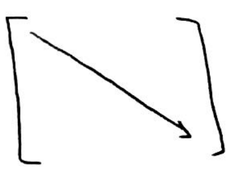
$T_{\mu}{}^{\mu}$ $\qquad$ Rank 0 $\;(0,0)$

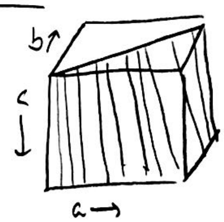
Tensor Contraction

$$\begin{matrix}(2,3)\\(1,0)\end{matrix} \rightarrow (2,2)$$

contra $\cdot$ cov $\rightarrow$ #

*Diagonal metric $\Rightarrow$ trace





Why do we care?

- Physics (units)
- Math (diff objects)
- Less Confusion

"A tensor is what transforms like a tensor" X

Thus, "matrix"

- $\vec{v} \rightarrow \vec{v}$ $\quad (1,1)$ $\qquad$ map (why det on metric weird)
- $\vec{v}, \vec{v} \rightarrow$ # $\quad (0,2)$ $\qquad$ quadratic form
- "mistake" with metric tensor $\quad M_{\mu\nu}$ v $M^{\mu}{}_{\nu}$